J. Fluid Mech. (1974), vol. 66, part 3, pp. 529–539
Printed in Great Britain

Non-Newtonian secretion flow in tubes

By J. M. FITZ-GERALD
Department of Mathematics, University of Queensland, Brisbane, Australia

(Received 23 April 1974)

A model is developed for the phenomenon of non-Newtonian secretion in tubes. The motivation for this study is the problem of glandular secretion, particularly in the pancreas. Power-law fluids are considered in some detail, as being biologically appropriate. It is found that for a power-law fluid whose exponent is less than unity (the biological case) two types of flow occur. For a sufficiently high secretion pressure, all of the tube is used for secretion, and a nonlinear pressure profile results. Numerical solutions are obtained for the pressure and rate of efflux. When the secretion pressure parameter falls below a certain critical value, the upper end of the tube begins to be choked off, only part of the tube being used for secretion. This phenomenon does not occur for exponents greater than or equal to unity. Physiological implications are considered, and a qualitative discussion given for the case of non-power-law fluids.

1. Introduction

In a recent paper (Heike & Fitz-Gerald 1974), the problem of reduction of the pancreatic secretion rate by an increase in the viscosity of duodenal aspirate (the secreted fluid) was considered. The model involved secretion (by filtration under external pressure) of a power-law fluid into a long tube, closed at one end, with subsequent ejection from the open end under the action of a pressure gradient set up by the secretion pressure; inertial forces were ignored. The purpose there was to gain qualitative information about the effect of a viscosity increase on the flow rate under pathological conditions, to assist in the use of viscosity measurements as a diagnostic tool. Relationships between the viscosity and flow rate were derived, within ranges of the physical and physiological parameters appropriate to the pancreas.

Further analysis shows that the phenomenon of secretion flow is more complex than was at first thought. In view of the possible application to a large number of secretion processes in biology, and the not inconsiderable fluid dynamic interest, this paper discusses the problem in some detail. A range of power-law fluids is considered, rather than the square-root-law fluid assumed for pancreatic secretion; further, solutions are obtained for an extensive range of the non-dimensional parameter G which governs the viscosity–flux relation. G is essentially the ratio of the secretion pressure available and a typical viscous force. Several interesting features emerge.

For large values of G, the pressure is virtually equal to the downstream

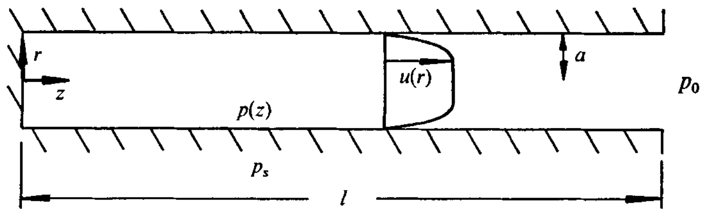

FIGURE 1. Dimensions and co-ordinates for the model secreting duct.

reference value everywhere, and the efflux tends asymptotically to a maximum value. As G decreases, the upstream pressure gradually increases, tending to its limiting value (the secretion pressure), while the efflux decreases. For fluids whose viscosity decreases with increasing shear ('hypo-Newtonian' or pseudoplastic), this maximum upstream pressure is reached for a non-zero value of G. When G falls below the cut-off point, the flow becomes progressively choked off from the upstream end; the effective length of the tube, in which secretion actually occurs, is reduced, with the upstream portion remaining at the secretion pressure and providing no contribution to the efflux. In this range of G, which extends down to zero, the efflux decreases linearly with G.

This choking effect does not occur with fluids whose viscosity increases with increasing shear (hyper-Newtonian or dilatant). The upstream pressure tends to its maximum value as G tends to zero, with a corresponding smooth decrease in efflux.

2. The mathematical model

As in the previous paper, we consider secretion into a circular tube of length l and radius a, under a secretion pressure p_s; the pressure at the open end is taken to be the (zero) reference level. The filtration coefficient is α, i.e. the secretion rate q per unit area is $\alpha(p_s - p)$, where p is the pressure at any point just inside the tube.

Modelling a gland such as the pancreas by a single tube will of necessity lead to a greatly reduced surface area available for secretion, since the successively branched tubules are being ignored for the hydrodynamic discussion. This may be compensated for, however, by using an enhanced filtration coefficient α. Further, it is probable that α will depend on the fluid viscosity; it will, however, be constant for a given fluid. Effects due to a viscosity-dependent α will be considered later in the paper. Cylindrical co-ordinates are used, with the origin centred at the closed end (see figure 1).

To facilitate the analysis, we assume that the flow is quasi-parallel almost everywhere. This effectively requires the secretion rate at any point to be much smaller than the total flux there. The validity of this approximation depends on the solution of the mathematical model; we show *a posteriori* that excellent accuracy is guaranteed for sufficiently long tubes. As in lubrication theory, this ensures that inertial forces are negligible; we may therefore neglect the convective terms even though a modified Reynolds number is not necessarily small. Further, the only non-negligible shear component will be τ_{rz}, the axial drag due to radial velocity variation. In this model, we consider a power-law fluid, so that

$$\tau_{rz} = -K\,|\partial u/\partial r|^{\gamma}; \tag{1}$$

in a later section, however, the results obtained will be shown to be qualitatively applicable to a wide range of non-Newtonian fluids. Treatments of simple axial flow of a power-law fluid in a pipe under a constant pressure gradient have appeared previously (e.g. Metzner & Reed 1955; Bird 1956), with the aim of quantifying an effective Reynolds number, usually via the Fanning friction factor. The early part of the present analysis is similar in spirit to these discussions; however, it seems preferable to develop the theory here from first principles, in view of the considerable differences introduced by the driving secretion pressure.

The equation of conservation of momentum for steady flow may be written, in the usual notation, as

$$\mathbf{u}\,.\,\nabla\mathbf{u}+\rho^{-1}\nabla p = \rho^{-1}\nabla\,.\,\boldsymbol{\tau}. \tag{2}$$

Non-dimensional variables are introduced to facilitate the analysis. A scale velocity is required; this will be taken as the mean exit velocity U_s which would be produced were the whole tube wall area (excluding the closed end) secreting under the maximum available pressure p_s. Since the secretion rate q per unit area is $\alpha(p_s-p)$, where p is the pressure at any point just inside the wall, U_s may easily be shown to be given by

$$U_s = 2l\alpha p_s/a. \tag{3}$$

We may now define

$$\left.\begin{aligned} r^* &= r/a, \qquad z^* = z/a,\\ \mathbf{u}^* &= \mathbf{u}/U_s, \quad p^* = p/p_s,\\ L &= l/a, \qquad \kappa = K/\rho,\\ q^* &= \alpha(p_s-p)/U_s = (1-p^*)/2L, \end{aligned}\right\} \tag{4}$$

and drop the asterisks for convenience; unless otherwise stated, non-dimensional quantities will be used from now on.

Equation (2) now becomes

$$F\mathbf{u}\,.\,\nabla\mathbf{u}+G\nabla p = \nabla\,.\,\boldsymbol{\tau}, \tag{5}$$

where

$$F = \frac{U_s^{2-\gamma}a^{\gamma}}{\kappa}, \quad G = \frac{p_s}{K(U_s/a)^{\gamma}}. \tag{6), (7}$$

F is the parameter corresponding to a Reynolds number in Newtonian flow; while we may expect F to be small for some secretion processes, we shall see later that in the pancreas F normally takes values of the order of 10. G expresses the relative magnitude of a typical secretion pressure force and viscous force, and is the principal controlling parameter for the flow. In pancreatic secretion, G normally takes values of the order of 10^5.

As mentioned above, we now make the parallel-flow approximation, expecting the local secretion velocity (radial) to be much less than the mean axial velocity. Also, axial variations in u will be small compared with radial gradients. We are, in fact, assuming

$$\frac{\partial u}{\partial z}, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial z} \ll \frac{\partial u}{\partial r},$$

and that the convective terms are negligible. Essentially this is the lubrication assumption, implying that secretion will affect conservation of mass only, with a negligible effect on local momentum balance. This implies that p is a function of z only, and (5) may now be written as

$$G\frac{dp}{dz} = -\frac{1}{r}\frac{\partial}{\partial r}\left(r\left|\frac{\partial u}{\partial r}\right|^{\gamma}\right), \tag{8}$$

the minus sign occurring since $\partial u/\partial r$ will be everywhere negative (or zero, on the axis). Equation (8) may be integrated twice with respect to r, using the no-slip condition, and we obtain

$$u = \left(\frac{G}{2}\left|\frac{dp}{dz}\right|\right)^{1/\gamma}\frac{1}{\xi}(1-r^{\xi}), \tag{9}$$

where $\xi = (1+\gamma)/\gamma$. Equation (9) gives the local velocity profile in terms of the local pressure gradient. As in lubrication theory, the pressure gradient is now determined by a continuity equation involving the total flux past any tube cross-section.

At any z, this flux is given by

$$Q = \int_0^1 2\pi r u(r)\,dr = \frac{\pi}{\xi+2}\left(\frac{G}{2}\left|\frac{dp}{dz}\right|\right)^{1/\gamma}. \tag{10}$$

This must be equal to the total amount of fluid secreted per unit time through that part of the wall upstream of the chosen z:

$$Q = 2\pi\int_0^z q(z)\,dz = \frac{\pi}{L}\int_0^z (1-p)\,dz. \tag{11}$$

Equations (10) and (11) now give the continuity equation in the form

$$\frac{1}{\xi+2}\left(\frac{G}{2}\left|\frac{dp}{dz}\right|\right)^{1/\gamma} = \frac{1}{L}\int_0^z (1-p)\,dz. \tag{12}$$

This is essentially a second-order nonlinear integro-differential equation for p.

Since dp/dz is everywhere negative, we may set $P(z) = 1-p$, and (12) becomes

$$\lambda\left(\frac{dP}{dz}\right)^{1/\gamma} = \int_0^z P(z)\,dz, \tag{13}$$

where

$$\lambda = \frac{L}{\xi+2}\left(\frac{G}{2}\right)^{1/\gamma}. \tag{14}$$

Two boundary conditions are required for P. Clearly, since the open end of the tube is exposed to the reference pressure, $p = 0$ at $z = L$, i.e. $P = 1$ there. Further, (13) shows that $dP/dz = 0$ at $z = 0$. We therefore specify

$$P(L) = 1, \quad P'(0) = 0. \tag{15a,b}$$

One differentiation of (13) gives

$$P''(P')^{(1-\gamma)/\gamma} - (\gamma/\lambda)P = 0; \tag{16}$$

and using the substitution

$$S = P'^2 \tag{17}$$

(16) now becomes

$$\frac{dS}{dP} S^{(1-\gamma)/2\gamma} - \frac{2\gamma}{\lambda} P = 0,$$

which is separable. In fact we find

$$S = \left[\frac{1+\gamma}{2\lambda}(P^2-\beta^2)\right]^{2\gamma/(1+\gamma)}, \tag{18}$$

where

$$\beta = P(0) \tag{19}$$

may lie in the range $0 \leqslant \beta \leqslant 1$ and is to be determined using (15*a*). Writing (18) in the form

$$\frac{dP}{dz} = \left[\frac{1+\gamma}{2\lambda}(P^2-\beta^2)\right]^{\gamma/(\gamma+1)}, \tag{20}$$

we may now integrate directly to obtain

$$z = \left(\frac{2\lambda}{1+\gamma}\right)^{\gamma/(1+\gamma)} \beta^{(1-\gamma)/(1+\gamma)} \int_0^{\phi} \sinh^{(1-\gamma)/(1+\gamma)} \theta \, d\theta, \tag{21}$$

with $\phi = \ln\{[P+(P^2-\beta^2)^{\frac{1}{2}}]/\beta\}$,

where β is determined from

$$L = \left(\frac{2\lambda}{1+\gamma}\right)^{\gamma/(1+\gamma)} \beta^{(1-\gamma)/(1+\gamma)} \int_0^{\psi} \sinh^{(1-\gamma)/(1+\gamma)} \theta \, d\theta, \tag{22}$$

with $\psi = \ln\{[1+(1-\beta^2)^{\frac{1}{2}}]/\beta\}$.

The outflow Q_E from the tube may now easily be obtained from (10) and (20):

$$Q_E = \pi\left[\frac{G(1+\gamma)(1-\beta^2)}{4L(\xi+2)^{\gamma}}\right]^{1/(1+\gamma)}. \tag{23}$$

This completes the required solution. Efflux and pressure profiles may now be obtained for a range of values of G and γ numerically. In limiting cases, however, asymptotic forms are available.

3. Solution for large G

When G is large, and viscous forces are thus relatively small, we expect the solution to tend to the limiting case $P = 1$ ($p = 0$) everywhere. This implies that β is very nearly unity and that Q_E is close to its maximum value π. (Recall that the flux has been scaled with respect to the mean velocity occurring when secretion occurs everywhere at the maximum rate; the maximum dimensional flux is therefore $\pi a^2 U_s$.) We therefore set $\beta = 1-\epsilon$, $\epsilon \ll 1$, in (22);

$$\ln\{[1-(1-\beta^2)^{\frac{1}{2}}]/\beta\} \sim (2\epsilon)^{\frac{1}{2}},$$

and in this range $\sinh\theta \sim \theta$. Equation (22) then becomes

$$L \sim \left(\frac{2\lambda}{\gamma+1}\right)^{\gamma/(1+\gamma)} (1-\epsilon)^{(1-\gamma)/(1+\gamma)} \int_0^{(2\epsilon)^{\frac{1}{2}}} \theta^{(1-\gamma)/(1+\gamma)} \, d\theta,$$

and this reduces to

$$B \sim 1-2L(\xi+2)^{\gamma}/G(1+\gamma), \quad Q_E \sim \pi. \tag{24}$$

It will be of interest in the later discussion on physiological implications to know how Q_E decreases with decreasing G. Asymptotic analysis to the next order of accuracy gives, after some tedious but straightforward manipulation,

$$\beta \sim 1-\frac{(\xi+2)^{\gamma}}{1+\gamma}\left(\frac{2L}{G}\right)+\frac{(\xi+2)^{2\gamma}(6\gamma^2-\gamma+10)}{6(\gamma+1)^2(2+\gamma)}\left(\frac{2L}{G}\right)^2$$

and

$$Q_E \sim \pi\left[1-\frac{(\xi+2)^{\gamma}(8+\gamma+3\gamma^2)}{3(1+\gamma)(2+\gamma)}\frac{2L}{G}\right]. \tag{24a}$$

4. Solution for minimum G

In the previous asymptotic analysis, γ could take any positive value. Now, however, the form of the 'small G' solution depends on the value of γ; for $\gamma < 1$, the integral in (22) is dominated by the behaviour near the upper limit as $\beta \to 0$, while this is not so for $\gamma > 1$. We now consider the $\gamma < 1$ case separately.

As G decreases, we expect β to decrease towards the limiting value zero ($P(0) \to 0$, i.e. $p(0) \to 1$). This suggests estimating a solution of (22) asymptotically for $\beta \ll 1$. In this case,

$$\ln\{[1+(1-\beta^2)^{\frac{1}{2}}]/\beta\} \to \ln(2/\beta),$$

and as $\beta \to 0$, the integral is dominated by the upper end, where

$$\sinh^{(1-\gamma)/(1+\gamma)}\theta \sim (\tfrac{1}{2})^{(1-\gamma)/(1+\gamma)}\exp[(1-\gamma)\theta/(1+\gamma)].$$

Equation (22) then becomes

$$L \sim \left(\frac{2\lambda}{\gamma+1}\right)^{\gamma/(1+\gamma)}\frac{\beta^{(1-\gamma)/(1+\gamma)}}{2^{(1-\gamma)/(1+\gamma)}}\int_0^{\ln(2/\beta)}\exp[(1-\gamma)\theta/(1+\gamma)]\,d\theta$$

and this reduces to

$$L \sim \frac{\lambda^{\gamma(1+\gamma)}(1+\gamma)^{(1-\gamma)/(1+\gamma)}}{2^{\gamma/(1+\gamma)}(1-\gamma)}.$$

This implies that the limiting value $\beta = 0$ is achieved when

$$\frac{G}{2L} = \frac{G_{\lim}}{2L} = \frac{(1-\gamma)^{1+\gamma}(\xi+2)^{\gamma}}{2^{\gamma}(1+\gamma)^{1-\gamma}} \tag{25}$$

and

$$Q_E = \pi[\tfrac{1}{2}(1-\gamma)(1+\gamma)^{\gamma/1+\gamma)}]; \tag{26}$$

no solution in this form is available when G takes smaller values. We denote the corresponding pressure profile by $P_{\lim}(z;\gamma)$.

On the other hand, the integral in (22) converges for $\beta \to 0$ when $\gamma > 1$, since $\sinh^{(1-\gamma)/(1+\gamma)}\theta \sim (\frac{1}{2}e^{\theta})^{-(\gamma-1)/(\gamma+1)}$ as $\theta \to \infty$ and $\sim \theta^{-(\gamma-1)/(\gamma+1)}$ as $\theta \to 0$, both of which give finite contributions near the limiting end points. Equation (22) becomes in this case

$$L \sim \left(\frac{2\lambda}{1+\gamma}\right)^{\gamma/(1-\gamma)}\beta^{(1-\gamma)(1+\gamma)}I_{\gamma},$$

where

$$I_{\gamma} = \int_0^{\infty}\sinh^{(1-\gamma)/(1+\gamma)}\theta\,d\theta,$$

and clearly, since this implies that

$$\beta = \left(\frac{2I_\gamma^{(1+\gamma)/\gamma}}{(1+\gamma)(\xi+2)}\right)^{\gamma/(\gamma-1)}\left(\frac{G}{2L}\right)^{1/(\gamma-1)},$$

$\beta \to 0$ as $G \to 0$, and a solution of this form is available for all values of G.

Finally, we note that, when $\gamma = 1$ (Newtonian fluid), (16) may be solved exactly; (16) becomes, in fact,

$$P'' - \lambda^{-1}P = 0,$$

with solution

$$P = \cosh(z/\lambda^{\frac{1}{2}})/\cosh(L/\lambda^{\frac{1}{2}}),$$

satisfying the boundary conditions at $z = 0$ and L. Clearly, solutions are available for all values of G in this case also.

5. The choked-flow case

We now consider the interesting case of a fluid with $\gamma < 1$ whose viscosity is so high that G falls below G_{lim}. The parameter β cannot be allowed to be negative, since this implies a pressure developed in the tube greater than the driving secretion pressure. However, we note that the 'trivial' solution $P = 0$ (i.e. $p = 1$) is also available for (16). Further, an acceptable solution in the previous form exists for a smaller effective value of L, say L', where

$$L' = \frac{2^{\gamma-1}(1+\gamma)^{1-\gamma}G}{(1-\gamma)^{1+\gamma}(\xi+2)^{\gamma}},$$

from (25). The solution we require, then, is

$$P(z) = \begin{cases} 0 & (0 \leqslant z \leqslant L-L'), \\ P_{\mathrm{lim}}(z-L+L',\gamma) & (L-L' \leqslant z \leqslant L). \end{cases}$$

We need only note that this is sufficiently continuous in the whole range to be an acceptable solution of (16).

Physically, this represents a situation where no secretion occurs in the region $0 \leqslant z \leqslant L-L'$, since no pressure difference exists across the wall. The upper part of the tube is effectively 'choked off' by the decreasing value of G (secretion pressure too small or viscous stresses too great). The non-dimensional flux Q_E remains at the value given by (26) for $G \leqslant G_{\mathrm{lim}}$, since $G/2L'$ will remain at its largest available value, given by (25). However, since the scale factor for the volume flux is $\pi a^2 U_s = 2\pi a^2 L\alpha p_s$, a decrease in the effective L results in a corresponding linear reduction in the physical volume outflow.

6. Numerical results

For given values of G, numerical quadrature techniques provide solutions for β, Q_E and p. Figure 2 shows some typical pressure profiles; figure 3 shows the dependence of the non-dimensional outflow Q_E on G for a range of values of γ. The transition from whole-tube to choked flow is demonstrated for $\gamma = 0{\cdot}5$ and $\gamma = 0{\cdot}3333\ldots$; no corresponding change occurs for $\gamma \geqslant 1$.

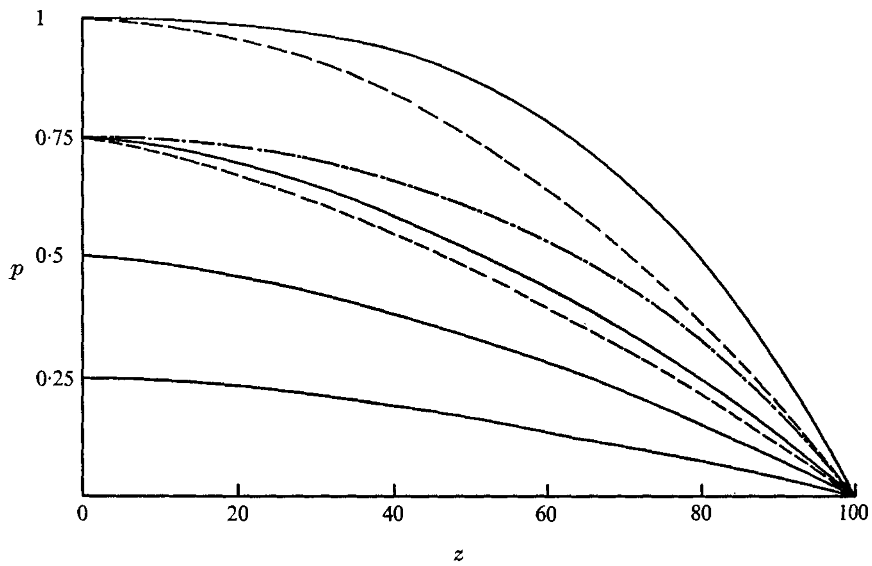

FIGURE 2. Typical pressure profiles in the duct, for $L = 100$ (in non-dimensional form). ——, $\gamma = 0{\cdot}5$; – – –, $\gamma = \frac{1}{3}$; —·—, $\gamma = 1{\cdot}0$.

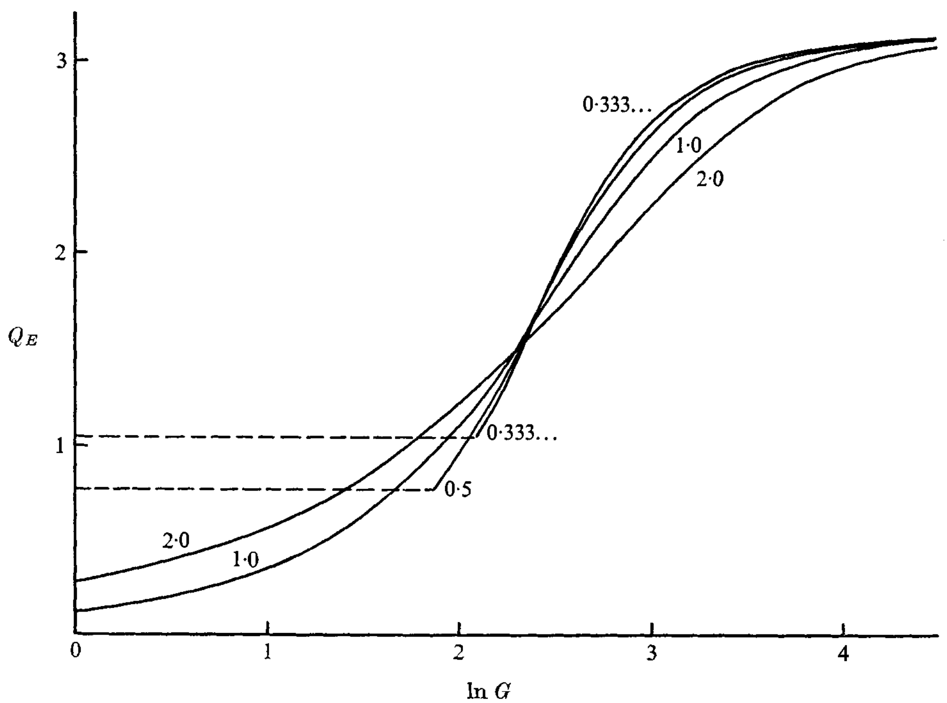

FIGURE 3. Non-dimensional efflux Q_E *vs.* secretion parameter G. – – –, choked-flow continuation for $\gamma < 1$ curves. Values of γ are shown on the figure.

7. Validity of the parallel-flow assumption

Essentially, this assumption requires that the rate of entry of fluid through the walls per unit length of tube be small compared with the total flux at any point. The rate of entry per unit length in non-dimensional terms is simply

$$2\pi q = \pi L^{-1}(1-p),$$

while the total flux Q is $\pi[\frac{1}{2}G|dp/dz|]^{1/\gamma}/(\xi+2)$, from (10). With $P = 1-p$, and making use of (19) to evaluate Q, we find that the ratio of the rate of entry to the flux, E say, is given by

$$E = \frac{\pi L^{-1}P}{\dfrac{\pi}{\xi+2}\left(\dfrac{G}{2}\right)^{1/\gamma}\left[\dfrac{1+\gamma}{2\lambda}(P^2-\beta^2)\right]^{1/(\gamma+1)}}.$$

Substitution for λ now gives

$$E = \frac{P}{L(P^2-\beta^2)^{1/(1+\gamma)}}\frac{(\xi+2)^{\gamma/(1+\gamma)}}{(1+\gamma)^{1/(1+\gamma)}}2^{1/(1+\gamma)}\frac{1}{(G/2L)^{1/(1+\gamma)}}, \tag{27}$$

the most convenient form.

Estimates of E may be given for both large and small G as limiting cases. For large G, both P and β are nearly unity; in this case

$$\frac{P}{(P^2-\beta^2)^{1/(1+\gamma)}} \sim \frac{1}{2^{1/(1+\gamma)}(P-\beta)^{1/(1+\gamma)}},$$

and we find that

$$E \sim \left[\frac{(\xi+2)^{\gamma}}{1+\gamma}\frac{1}{L^{1+\gamma}(G/2L)(P-\beta)}\right]^{1/(1+\gamma)}.$$

Thus E will certainly be small provided that

$$P-\beta \gg \frac{(\xi+2)^{\gamma}}{1+\gamma}\frac{1}{L^{1+\gamma}(G/2L)}; \tag{28}$$

and from (24), we find that the maximum value of $P-\beta$ (at $z = L$) is

$$(P-\beta)_{\max} = \frac{(\xi+2)^{\gamma}}{1+\gamma}\frac{1}{G/2L},$$

certainly very small for sufficiently large G. Then (28) becomes

$$P-\beta \gg L^{-(1+\gamma)}(P-\beta)_{\max}.$$

For suitably large L (length-to-radius ratio), therefore, E will be very small except in a small region near the closed end. For example, for $L = 100$ and $\gamma = \frac{1}{2}$ (typical values for pancreatic secretion), E is small, say $< 0{\cdot}1$, whenever $P-\beta$ is greater than about $0{\cdot}03$ of its maximum value.

When G is small, i.e. $\beta \sim 0$, (27) becomes

$$E \sim \left[\frac{2(\xi+2)^{\gamma}}{L^{1+\gamma}P^{1-\gamma}(1+\gamma)G/2L}\right]^{1/(1+\gamma)}. \tag{29}$$

If $\gamma < 1$, this occurs at the limiting value

$$\frac{G}{2L} = \frac{(1-\gamma)^{1+\gamma}}{1+\gamma}\frac{(\xi+2)^{\gamma}}{2^{\gamma}};$$

and now

$$E \sim \left[\frac{2}{L(1-\gamma)\,P^{(1-\gamma)/(1+\gamma)}}\right].$$

E will be sufficiently small to justify the parallel-flow assumption provided again that L is suitably large; and the larger the value of L, the greater the region of the flow in which the assumption is valid, recalling that P takes a maximum value of 1.

On the other hand, when $\gamma > 1$ the right-hand side of (29) will be small provided that

$$P^{\gamma-1} \ll \frac{(1+\gamma)\,L^{1+\gamma}(G/2L)}{2(\xi+2)^{\gamma}};$$

and once again, for large L, only very small values of G give rise to a flow in which the parallel-flow assumption is not valid.

8. Other non-Newtonian fluids

Consider a fluid which does not have power-law behaviour, but whose viscosity character can be 'bounded' by two power-law expressions:

$$\tau = -Kf(|\partial u/\partial r|),$$

where $ax^{\nu} \leqslant f(x) \leqslant bx^{\mu}$, say ($a$ and b constants). Further, suppose that ν and μ are both either greater or less than unity; i.e. the fluid is either 'hyper-Newtonian' or 'hypo-Newtonian' always. Let $h(x)$ be the inverse function of f, i.e.

$$x = h(f(x)).$$

Then h will also be bounded by powers of x, both either greater or less than unity; but if the exponents for f are less than unity, those for h are greater, and vice versa.

If we consider the analysis which produces the dichotomy of behaviour for hyper-Newtonian and hypo-Newtonian power-law fluids, we see that the distinction may be drawn from (16); if $(1-\gamma)/\gamma$ is positive, then when the integration yielding (18) is performed, we find $2\gamma/(1+\gamma) > 1$, leading to the infinite integral in (22) as $\beta \to 0$, and the limiting value for G. Now consider the fluid described above. A derivation of the equation for the pressure gradient exactly analogous to the previous one yields an equation analogous to (16):

$$\frac{P''}{P'}\left[h\left(\frac{G}{2}P'\right) - \int_0^1 2r^2 h\left(r\frac{G}{2}P'\right)dr - \int_0^1 2r\int_r^1 h\left(\rho\frac{G}{2}P'\right)d\rho\,dr\right] = \frac{P}{L}. \quad (30)$$

Now, if $m\left(\frac{G}{2}P'\right)^{1/\mu} \leqslant h\left(\frac{G}{2}P'\right) \leqslant n\left(\frac{G}{2}P'\right)^{1/\nu}$, then the same is true of the expression in square brackets above. Hence, for a hypo-Newtonian fluid, (30) will be bounded by equations of the form

$$P''(P')^{(1-\mu)/\mu} = \frac{k_1}{LG^{1/\mu}}P$$

and

$$P''(P')^{(1-\nu)/\nu} = \frac{k_2}{LG^{1/\nu}}P,$$

where $(1-\mu)/\mu$ and $(1-\nu)/\nu$ are both positive. The resulting solution will therefore have the choked-flow character found earlier. A similar argument holds for the hyper-Newtonian case.

9. Physiological implications

For the pancreas, K is normally about $0{\cdot}02\,\mathrm{cm}^2/\mathrm{s}$, γ is about $0{\cdot}5$, L of the order of 100, and G is of the order of 10^5; the exact value is uncertain because of difficulties in estimating α. However, we can still estimate the effect of an increase in viscosity under pathological conditions. Using the large G approximation, we find from (24a) that Q_E is virtually constant for a considerable range of G. With the values given, (24a) becomes

$$Q_E \sim \pi[1-370/G],$$

so that even an extreme reduction in G by a factor of 100 causes a reduction in the non-dimensional efflux of the order of 30 %. As pointed out by Heike (1973), considerable reductions in efflux certainly occur in fibrocystic patients, so that some effect not reproduced by the current model must be responsible; the model does not provide sufficient information to allow viscometry to be used as a convenient diagnostic tool.

We expect that a reduction in α will occur if there is an increase in viscosity; typically $\alpha \propto \mu^{-1}$ for Newtonian fluids in filters such as porous bronze. It is not clear in the pseudoplastic situation what the appropriate viscosity coefficient is; presumably large shear rates occur in the pores of the endothelium of the ductules, and some form of limiting high-shear viscosity coefficient will govern the filtration rate.

Other possible geometric factors influencing the physical outflow rate are discussed in the previous paper (Heike & Fitz-Gerald 1974); inclusion of these in the model would be necessary before their effects could be assessed. While values of G seem too large for the choking effect to occur in the pancreas, it is possible that local reductions in secretion pressure, perhaps combined with the very small radii for the finest ductules, may well lead to a reduction in efficiency as parts of the gland are rendered inoperative as progressive choking occurs.

REFERENCES

Bird, R. B. 1956 *A.I.Ch.E. J.* **2**, 428.
Heike, J. 1973 Thesis, University of Queensland.
Heike, J. & Fitz-Gerald, J. M. 1974 *Biorheology*, to appear.
Metzner, A. B. & Reed, J. C. 1955 *A.I.Ch.E. J.* **1**, 434.